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AN INVESTIGATION OF ELASTIC PARABOLOIDAL SHELLS OF REVOLUTION

A THESIS

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LIST OF SYMBOLS

Symbol	Definition
X, Y, Z	Rectangular coordinates
h	Thickness of a shell
p	Uniform pressure per unit area
u, v, w	Components of displacements in the direction in the radial, tangential and axial directions, respectively
φ, ξ	Unit elongations of a shell in meridianal direction and in the direction of parallel circle, respectively
E	Modulus of elasticity in tension and compression Poisson's ratio
D	Flexural rigidity of a plate or $D = \dfrac{Eh^3}{12\,(1 - v^2)}$
r_1, r_2	Radii of curvature of a shell in the form of a surface of revolution in meridianal plane and in the normal plane perpendicular to meridian, respectively
r	Radii of circle of a shell in the form of a surface of revolution
$N\varphi, N\xi$	Membrane forces per unit length of principal normal sections of a shell
$M\varphi, M\xi$	Bending moments in a shell per unit length of meridianal section and section perpendicular to meridian, respectively
$Q\varphi, Q\xi$	Shearing forces parallel to Y-axis per unit length of an axial section and a section perpendicular to the axis of a cylindrical shell, respectively
H, V	Horizontal and vertical stress resultants related to Q and N, respectively
β	The angle enclosed by the tangents to deformed meridian, at one and the same material point

Symbol	Definition
φ	The angle made by the normal to the middle surface of shells of revolution and the axis of rotation
ξ	The angle made by the radii r of circle of the middle surface of shells of revolution and the X-axis
P_H, P_V	The components of load intensity in the r and Y directions, respectively

SUMMARY

The purpose of this study is to derive a system of equations which can be used for the analysis of stress distribution as well as the deformation of thin elastic paraboloidal shells of revolution under axially symmetrical loads. This set of canonical equations is valid for paraboloidal shells of uniform thickness as well as a large class of paraboloidal shells of non-uniform thickness.

In order to gain the advantage of being able to pass directly from the equations of shell theory to the equations of stretching and bending of circular plates, according to E. Reissner [3], the two simultaneous second-order differential equations are obtained by choosing as one of the two basic variables the quantity rH rather than r_2Q, where r is the radius of a parallel circle of a shell in the form of a surface of revolution and H is the horizontal stress resultant; r_2 is the radius of the principal curvature of a shell in the form of a surface of revolution in meridianal plane and Q is the radial shearing forces. According to Naghdi and DeSilva [4] these two simultaneous second-order differential equations can also be combined into a single second-order complex differential equation. In this study, by substituting the properties of paraboloidal shells into this complex differential equation, it is reduced to a canonical equation of paraboloidal shells of revolution.

By means of a method of asymptotic integration developed by Langer [5], the homogeneous solution of the complex differential equation is obtained. This solution is valid at the apex of the shell where a second-

order pole is present in differential equation and involves Kelvin functions. The solution obtained according to the membrane theory of shells given by Timoshenko [6] is used for the particular solution. With these two solutions, the particular problems of paraboloidal shells can be solved by the method of superposition.

In this study, the analysis of stress distribution and deformation of the paraboloidal shell of revolution of constant thickness under uniform edge loads has specially been investigated in detail. Also included is the comparison of the solution with the equivalent shallow spherical shell (their radii of principal curvature are approximately equal in a small angle region).

CHAPTER I

INTRODUCTION

An essential step in the treatment of the small deflection problems of shells of revolution is the reduction of all the elasticity equations into two simultaneous second-order differential equations. This reduction was first given by H. Reissner [1] for spherical shells of constant thickness. Subsequently E. Meissner [2] published the corresponding reduction for general shells of revolution.

A formulation of finite deformation theory of shells of revolution, containing the theory of small deflection (linear theory) as well as the historical development of the subject was discussed by E. Reissner [3]. Naghdi and De Silva [4] has shown that the resulting differential equations for small deformation in thin elastic shells of revolution, derived by E. Reissner, may be combined into a single second-order complex differential equation. This differential equation is valid for shells of uniform thickness and a large class of shells of non-uniform thickness.

The analysis of stress distribution and deformation of paraboloidal shells of revolution of uniform thickness under axisymmetric loads based upon E. Reissner's theory is investigated in this study. The Reissner's theory is discussed in this study. Solutions of shells closed at the apexes $\varphi = 0$ under edge loadings only are obtained.

The solution of the governing complex differential equation mentioned is obtained by means of a method of asymptotic integration due to

Langer [5]. This solution is valid at the apex of the shell where a second-order pole is present in the differential equation. Bessel functions of complex argument are involved in the solution.

The bending solutions of the shells under uniform edge loads can be super-imposed on the stress distribution and displacement obtained according to membrane analysis. Thus, the resulting solution will give the stress distribution and displacement of paraboloidal shells of revolution under exisymmetrical load with any type of edge constraints.

A comparison of the results is made between a shallow paraboloidal shell of revolution according to the method used in the study and a shallow spherical shell according to the approximate solution discussed by Timoshenko [6].

CHAPTER II

BASIC THEORY AND EQUATIONS

Review of Reissner's Theory and Basic Equations for General Shells of Revolution

a. Geometrical relations

Let X, Y, and Z represent a set of cartesian coordinates, φ and ξ be a set of parametric coordinates embedded in the undeformed middle surface of the shell as shown in Figure 1. If r represents the radius of the parallel circles defined by the parameter φ, the equation of the middle surface of the shell may be expressed in the following parametric form:

$$r = r(\varphi) \tag{1}$$

$$Y = Y(\varphi) \tag{2}$$

The sloping angle ϕ of the tangents to a meridian curve is given by

$$\tan \phi = dY/dr \tag{3}$$

From Equation (3) it follows that

$$\cos \phi = r'/\gamma \tag{4}$$

$$\sin \phi = Y'/\gamma \tag{5}$$

where γ is given in the form

$$\gamma = \left[(r')^2 + (Y')^2\right]^{1/2} \tag{6}$$

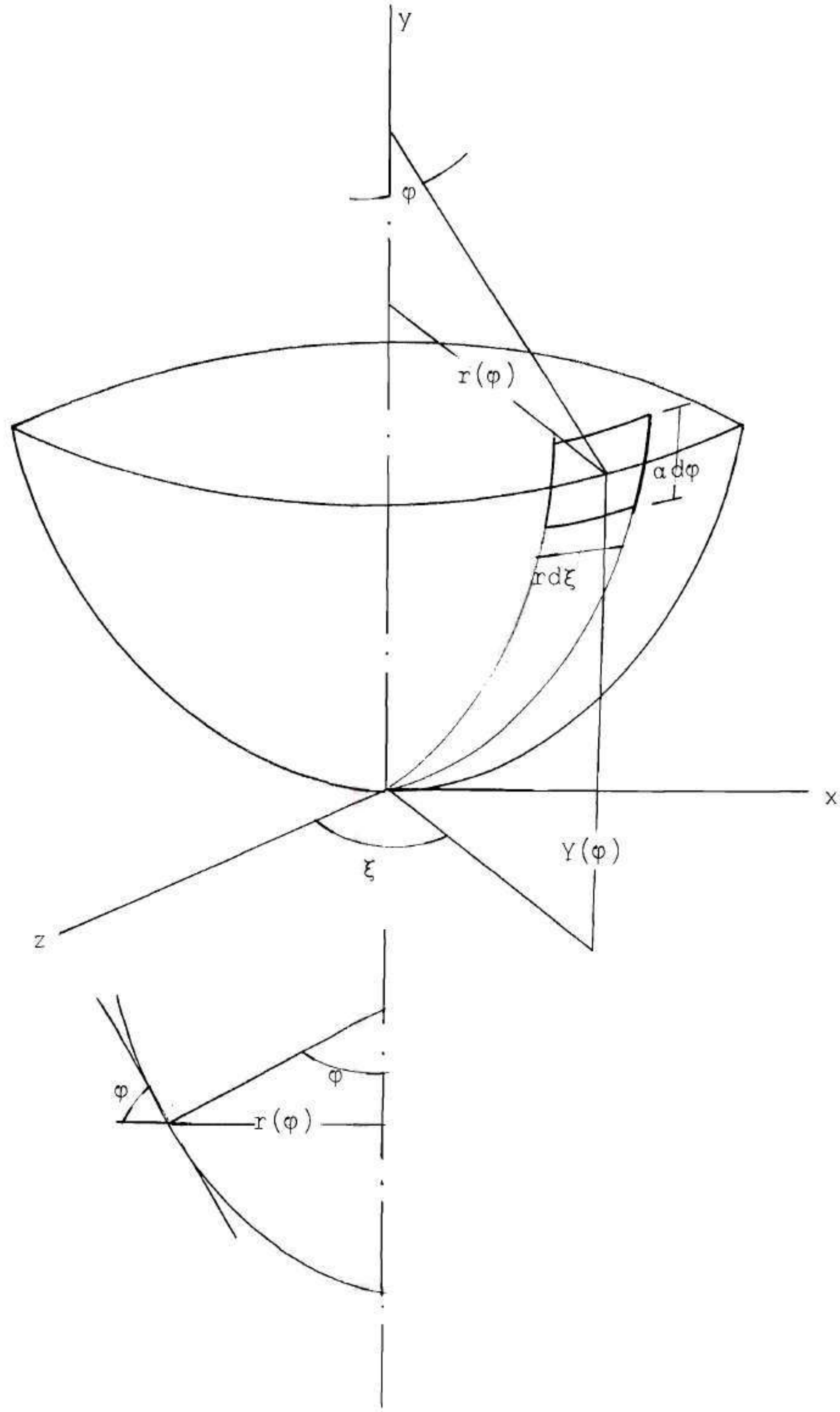


Figure 1. Orthogonal Coordinates φ, ξ, on Middle Surface of a Shell of Revolution.

It is noted in [8] that

$$1/r_2 = \frac{-1}{[r^2(1 + r'^2)]^{1/2}} \tag{7}$$

$$1/r_1 = \frac{r''}{[(1 + r'^2)]^{3/2}} \tag{8}$$

where r_1 and r_2 are the principal radii of curvature of the middle surface of the shell as shown in Figure 2.

It follows from the geometry of the middle surface that

$$r = r_2 \sin \varphi \tag{9}$$

$$r = r_1 \tag{10}$$

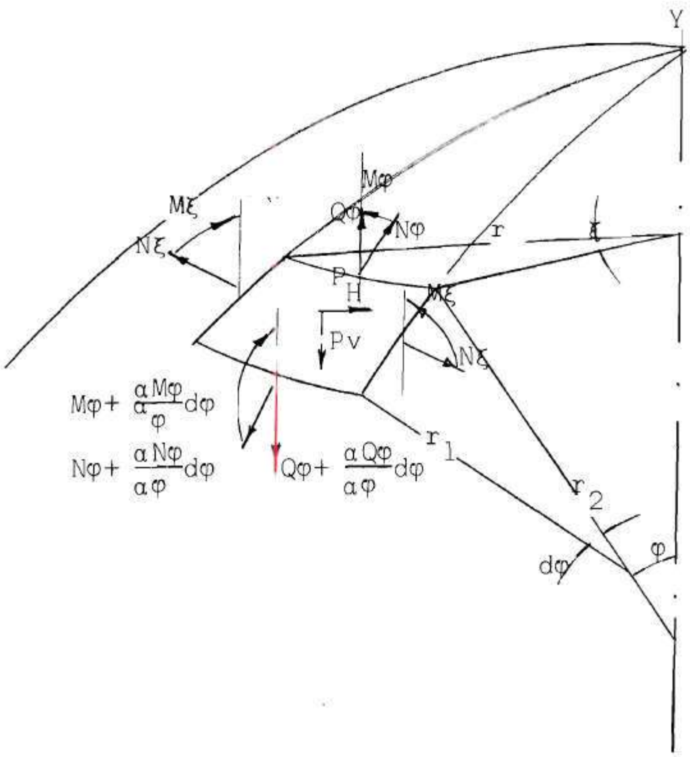


Figure 2. Element of Shell, Showing Components of Visible Stress Resultants, Stress Couples and Load Intensity.

b. Differential equations of compatibility and equilibrium

Based upon the assumptions that the normal to the Undeformed middle surface is deformed without extension into the normal to the deformed middle surface and the thickness h is small compared with the magnitudes of the radii of curvature r_1 and r_2 as defined by Equation (7) and Equation (8), the compatibility relation may be written as

$$\cos\varphi\ (r\epsilon_\xi)' - \cos(\varphi+\beta)(r'\epsilon_\varphi) = r'[\cos(\varphi+\beta)-\cos\varphi] \quad (11)$$

where β is the angle enclosed by the tangents to the deformed and undeformed meridian, at one and the same material point as shown in Figure 3. Equation (11) is expanded into a power series of β as follows:

$$\cos\Phi(r\epsilon_\xi)' - (\cos\varphi + \beta\sin\varphi + \ldots\ldots)\ (r'\epsilon_\varphi)$$
$$= r'(\beta\sin\varphi - (1/2)\,\beta^2\cos\varphi + \ldots\ldots) \quad (12)$$

where ϵ_φ and ϵ_ξ are unit elongations of a shell in the meridianal direction and in the direction of parallel circle, respectively.

Force and moment equilibrium equations for elements of the shell are

$$(r\ V)' + rr_1 P_V = 0 \quad (13)$$

$$(r\ H)' - r_1 N_\xi + rr_1 P_H = 0 \quad (14)$$

$$(r\ M_\varphi)' - r_1\cos\varphi\ M_\xi + rr_1\ (H\sin\varphi - V\cos\varphi) = 0 \quad (15)$$

N_ξ, N_φ, and Q are the stress resultants; M_ξ and M_φ the stress couples as shown in Figure 2; H and V denote the "horizontal" and "vertical" stress resultants related to Q and $N\varphi$; P_v and P_H are the components of load intensity in the r and Y directions.

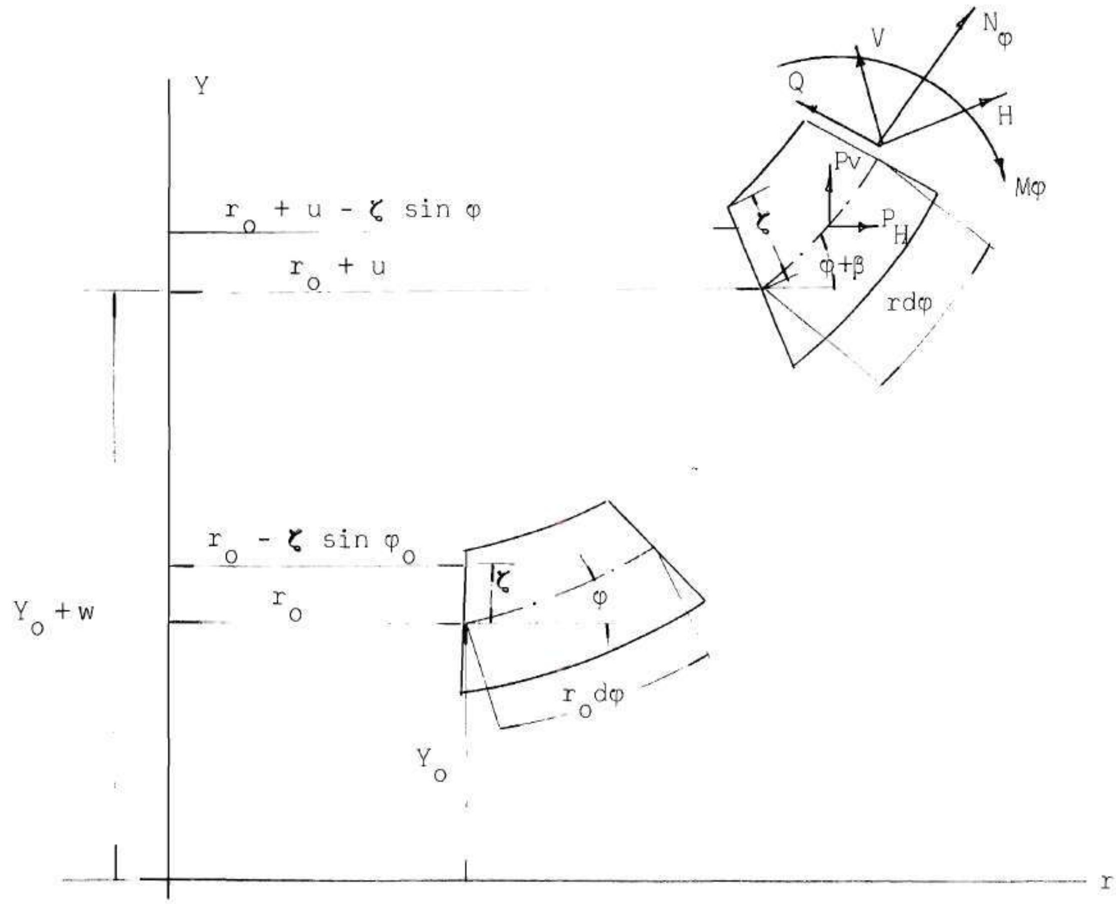


Figure 3. Deformation of a Shell Element, and the Components of the Visible Stress Resultants, Stress Couples and Load Intensity.

c. The basic equations of the small deflection theory of elastic shells of revolution under axisymmetric loading

Equations (13) and (14) may be rewritten as below:

$$r\,V = -\int r r_1\, P_v\, d\varphi \tag{16}$$

$$r_1 N\xi = (rH)' + rr_1 P_H \tag{17}$$

It is seen from Figure 3, the following relationships can be established

$$rN\varphi = (rH) \cos \varphi + (rV) \sin \varphi \tag{18}$$

$$rQ = -(rH) \sin \varphi + (rV) \cos \varphi \tag{19}$$

The moment-curvature relationships are

$$M\varphi = \frac{D}{r_1} [\beta' + \nu \frac{r'}{r} \beta] \tag{20}$$

$$M\xi = \frac{D}{r_1} [\frac{r'}{r} \beta + \nu\beta'] \tag{21}$$

The stress-displacement relationships are

$$u = \frac{r}{Eh} (N\xi - \nu N\varphi) \tag{22}$$

$$w = \int [\frac{Y'}{Eh} (N\varphi - \nu N\xi) - r'\beta] \, d\varphi \tag{23}$$

and the following stress-strain relationships for plane stress can be considered

$$\epsilon\varphi = \frac{1}{Eh} (N\varphi - \nu N\xi) \tag{24}$$

$$\epsilon\xi = \frac{1}{Eh} (N\xi - \nu N\varphi) \tag{25}$$

<u>d. Reduction of the equations of equilibrium and compatibility to two differential equations of the second-order</u>

The two simultaneous equations for β and rH are obtained by introducing $M\varphi$ and $M\xi$ from Equations (20) and (21) into the moment equilibrium equation (15) and by introducing $\epsilon\varphi$ and $\epsilon\xi$ from Equation (24) and Equation (25), with $N\varphi$ and $N\xi$ from Equations (17) and (18), into the compatibility equation (12). The results can be written as follows:

$$\beta'' + \frac{(rD/r_1)'}{(rD/r_1)}\beta' - [(\frac{r'}{r})^2 - \nu\frac{(r'D/r)'}{(rD/r)}]\beta + \frac{Y'}{(rD/r_1)}(rH)$$

$$= \frac{r'}{(rD/r_1)}(rV) \qquad (26)$$

$$(rH)'' + \frac{(r/r_1Eh)'}{(r/r_1Eh)}(rH)' - [(\frac{r'}{r})^2 - \nu\frac{(r'/r_1\ Eh)'}{(r/r_1\ Eh)}](rH)$$

$$- \frac{Y'}{(r/r_1Eh)}\beta = [\frac{r'Y'}{r^2} + \nu\frac{(Y'/r_1Eh)'}{(r/r_1\ Eh)}](rV)$$

$$+ \nu\frac{Y'}{r}(rV)' - [\frac{(r/r_1Eh)'}{(r/r_1Eh)} + \nu\frac{r'}{r}](rr_1P_H)$$

$$- (rr_1P_H)' \qquad (27)$$

where E is the Young's modulus, ν is the Poisson's ratio, and

$D = \dfrac{Eh^3}{12(1-\nu^2)}$ is the flexural rigidity of plate.

Reduction of the Two Simultaneous Differential Equations to a Single Complex Differential Equation

Naghdi and DeSilva have shown that the differential equations (26) and (27) may be combined into a single second order complex differential equation in a normal form, if suitable change of dependent variables can be made. The normal form of the differential equation is as follows:

$$Z'' + [2i^3 \mu^2 \chi^2(\varphi) + \Omega(\varphi)] Z =$$

$$[\frac{h}{h_o} \frac{r}{r_1}]^{1/2} (F + ikG) f(\varphi) \tag{28}$$

provided that k, given by [4]

$$k = -\frac{i}{2\mu^2} (\nu\lambda - \frac{\delta}{2}) + \left\{1 - [\frac{1}{2\mu^2} (\nu\lambda - \frac{\delta}{2})]^2\right\}^{1/2} \tag{29}$$

is either exactly or very nearly a constant.[1]

The various quantities occurring in Equations (28) and (29) are defined by

$$Z = (\frac{h}{h_o})^{3/2} (\frac{r}{r_1})^{1/2} (\beta + ik\chi) \tag{30}$$

$$i = \sqrt{-1}$$

$$\chi = \frac{m}{Eh^2} (rH) , \tag{31}$$

$$m = [12 (1 - \nu^2)]^{1/2} \tag{32}$$

$$\frac{r_1^2 m}{r_2 h} = 2\mu^2 f(\varphi) \tag{33}$$

$$\lambda = (\frac{h_o}{h} f(\varphi))^{-1} [\frac{(r'/r)'}{(r/r_1)} + 3 \frac{r'}{r} \frac{h'}{h}] \tag{34}$$

---

[1]Whenever the radius of curvature of the generating curve r_1 is a constant (r_2 may be a function of φ), then, with proper choice of φ, that k is in fact constant. The cases of conical shell and toroidal shell are included in this class.

$$\delta = 2\left(\frac{h_o}{h} f(\varphi)\right)^{-1} \cdot$$

$$\frac{h''}{h} + 2\,\frac{r'}{r}\,\frac{h'}{h} + \frac{(r/r_1)'}{(r/r_1)}\,\frac{h'}{h} \tag{35}$$

$$F = 2\mu^2 \frac{m}{Eh^2}\,(rV)\cot\varphi \tag{36}$$

$$G = \frac{h_o}{h} f(\varphi)^{-1}\left[\frac{Y'r'}{r^2} + \nu\,\frac{(Y'/r_1 Eh)'}{(r/r_1\,Eh)}\right]\zeta +$$

$$\nu\,\frac{Y'}{r}\,\zeta' - \left[\frac{(r/r_1\,Eh)'}{(r/r_1\,Eh)} + \nu\,\frac{r'}{r}\right] P_H - P_H' \tag{37}$$

and

$$\chi^2 = \left(k + i\,\frac{\nu\lambda}{2\mu^2}\right)\left(\frac{h_o}{h}\right) f(\varphi)$$

$$\Omega = -\frac{1}{2}\,\frac{(r/r_1)''}{(r/r_1)} + \frac{1}{4}\left\{\frac{(r/r_1)'}{(r/r_1)}\right\}^2 - \frac{(r')^2}{(r)^2} -$$

$$\frac{3}{2}\,\frac{(r/r_1)'}{(r/r_1)}\,\frac{h'}{h} - \frac{3}{2}\,\frac{h''}{h} - \frac{3}{4}\,\frac{(h')^2}{(h)^2} \tag{38}$$

$$\zeta = \frac{m}{Eh_o^2}\,(rV) \tag{39}$$

$$P_H = \frac{m}{Eh_o^2}\,r r_1 P_H \tag{40}$$

where μ is constant and it is to be noted that $f(\varphi)$ is independent of the thickness $h(\varphi)$; h_o is the value of h at some reference section (say $\varphi = \varphi_o$).

CHAPTER III

REDUCTION OF NAGHDI AND DE SILVA'S GENERAL SOLUTION TO PARABOLOIDAL SHELLS OF REVOLUTION

Basic Equation of Paraboloidal Shells of Revolution

For paraboloidal shells of revolution, the equation of the middle surface in rectangular Cartesian coordinates is given by

$$X^2 + Z^2 - b^2 = aY \tag{41}$$

It follows from Equation (7) and Equation (8) that the radii of curvature are

$$r_1 = \frac{A}{\cos^3 \varphi} \tag{42}$$

$$r_2 = \frac{A}{\cos \varphi} \tag{43}$$

$$r = A \tan \varphi \tag{44}$$

Substitution of Equations (42) and (43) into Equation (33) yields

$$f(\varphi) = \sec^5 \varphi \tag{45}$$

$$2\mu^2 = \frac{Am}{h} \tag{46}$$

where $A = \frac{a}{2}$, and when the thickness h is uniform, it is seen that

$$\delta = 0 \tag{47}$$

$$\lambda = - \cos^3 \varphi \tag{48}$$

$$\chi^2 = \sec^5 \varphi \tag{49}$$

Since h is very small when compared to r_1 and r_2, hence $\nu\lambda << 2\mu^2$ as the result of the assumption.

For paraboloidal shells r_1 is not constant, it follows that when h is uniform, k is a function of φ. However, with a view towards approximating k by a constant so that the condition for the validity of equation (28) is fulfilled, it also has been discussed above that $\nu\lambda << 2\mu^2$ and by Equation (29),

$$\frac{\nu\lambda}{2\mu^2} = 0 \quad \text{or} \quad k = 1 \tag{50}$$

With this approximation, equation (28) is valid and the coefficient function is

$$\Omega = 4\frac{1}{2} - \sec^2 \varphi + 2\frac{1}{4} \cot^2 \varphi - 3 \csc^2 \varphi \tag{51}$$

Substituting Equations (42) and (44) into Equation (30), Z becomes

$$Z = (\cos^2 \varphi \sin \varphi)^{1/2} (\beta + i\psi) \tag{52}$$

Solution of the Differential Equation by

Asymptotic Integration

Inspection of the coefficient function Ω as given by Equation (51) reveals the presence of a pole of second order (at $\varphi = 0$) in the differential Equation (28). Consequently, the solution of Equation (28) by the classical method of asymptotic integration fails to yield a solution valid at the apex of the shell ($\varphi = 0$).

The incidence of the Stokes' phenomenon [5] has been associated with the vanishing of the coefficient $\Omega(\varphi)$, and its quantitative aspects with the order to which that coefficient becomes zero. It is shown in [5] that this phenomenon is engendered also by an infinity in either of the coefficients, and is quantitatively dependent upon the structure of that infinity. Specifically when the coefficient $\Omega(\varphi)$ has a pole of second order, a number of standard differential equations may be brought under this general type. In order to obtain a solution of Equation (28) which is valid at $\varphi = 0$, recourse will be made to a technique of asymptotic integration, which has been discussed by Langer [5].

According to Langer's discussion Equation (28) must be modified by changing the independent variable as suggested by Naghdi and De Silva [9]. Let the new variable

$$t = \sin \frac{\varphi}{2} \tag{53}$$

$$dt = 2 \frac{dt}{(1 - t^2)^{1/2}} \tag{54}$$

Then Equation (28) may be written as

$$\frac{d^2Z}{dt^2} - \frac{t}{(1 - t^2)} \frac{dZ}{dt} + 4 \left[2i^3 \mu^2 \frac{\chi^2(t)}{(1 - t^2)} + \frac{\Omega(t)}{1 - t^2} \right]$$

$$= (1 - t^2)^{1/4} \frac{4}{(1 - t^2)^{5/4}} \left(\frac{r}{r_1}\right)^{1/2} (F + iG)\; f(t) \tag{55}$$

and Equation (51) becomes

$$\Omega(t) = 2\,\frac{1}{4} - \frac{3}{4}\,\frac{1}{4t^2(1 - t^2)} - \frac{1}{1 - 4t^2(1 - t^2)} \tag{56}$$

where $\Omega(t)$ is bounded in $|t| < 1$, i.e., in $0 \leq \varphi < \pi$. Now, by means of the transformation

$$\underline{Z} = (1 - t^2)^{1/2}\, Z \tag{57}$$

Equation (55) may be expressed as

$$\frac{d^2\underline{Z}}{dt^2} + [\frac{1}{2}\,(1 - t)^{-1} + \frac{3}{4}\,t^2\,(1 - t)^{-2} + \frac{2}{(1 - t^2)^{5/4}}$$

$$+ \frac{5t^2}{(1 - t^2)^{9/4}}]\,\underline{Z} = f(t)\,\frac{4}{(1 - t^2)}\,(\frac{r}{r_1})^{1/2}\,(F + iG) \tag{58}$$

Let

$$\Omega_1(t) = \frac{1}{2}\,(1 - t)^{-1} + \frac{3}{4}\,t^2\,(1 - t)^{-2} \tag{59}$$

substituting Equation (59) into Equation (58), gives

$$\frac{d^2\underline{Z}}{dt^2} + \left\{8i^3\,\mu^2\,\frac{\chi^2(t)}{(1 - t^2)} + \frac{4\Omega(t)}{(1 - t^2)} + \Omega_1(t)\right\}\,\underline{Z}$$

$$= f(t)\,\frac{4}{(1 - t^2)}\,(\frac{r}{r_1})^{1/2}\,(F + iG) \tag{60}$$

From Equation (56)

$$\frac{4\Omega(t)}{(1 - t)} = \frac{9}{1 - t^2} - \frac{3}{4}\,\frac{1}{t^2\,(1 - t^2)^2} - \frac{4}{(1 - t^2)[1-4t^2(1-t^2)]}$$

$$= -\,\frac{3}{4}\,\frac{1}{t^2} - \Omega_2(t) \tag{61}$$

where

$$\Omega_2(t) = \frac{3}{4}\frac{1}{t} + \frac{9}{1 - t^2} - \frac{3}{4}\frac{1}{t^2(1 - t^2)^2}$$

$$- \frac{4}{(1 - t^2)[1 - 4t^2(1 - t^2)]} \tag{62}$$

Expanding the coefficient function $\Omega_2(t)$ in Equation (62) by partial fractions, the term $1/t$ vanished and Equation (62) becomes

$$\Omega_2(t) = \frac{33}{4}\frac{1}{(1 - t^2)} - \frac{3}{4}\frac{1}{(1 - t^2)^2}$$

$$- \frac{4}{(1 - t)[1 - 4t^2\ (1 - t^2)]} \tag{63}$$

After substituting Equation (63) into Equation (60),

$$\frac{d^2\underline{Z}}{dt^2} + \left\{8i^3\ \mu^2 \frac{\chi^2(t)}{(1 - t^2)^{1/2}} + (\frac{-3/4}{t^2}) + \Omega_3(t)\right\}\ \underline{Z}$$

$$= f(t)\ \frac{4}{(1 - t^2)}\ (\frac{r}{r_1})^{1/2}\ (F + iG) \tag{64}$$

where

$$\Omega_3(t) = \Omega_1(t) + \Omega_2(t) \tag{65}$$

The numerator of the second term in the coefficient of $\underline{Z}$, when written as $1/4\ [1 - (2)^2]$ corresponds to $1/4\ (1 - A^2)$ in Langer's notation.[1]

---

[1]The normal form of the differential equation in (5) is

$$\frac{d^2u}{dz^2} + \left\{\rho^2\varphi^2(Z) + \frac{1/4 - A^2}{z^2} + \chi(\rho, Z)\right\}\ u = 0$$

where $-\pi/2 < \arg A \leqq \pi/2.$

Since the particular solution of Equation (64) depends on the specific loading, only the solution of the homogeneous differential equation associated with Equation (64) should be considered in this problem.

As pointed out by Langer, the homogeneous solutions of the differential equation of this type are

$$y_1(z) \equiv \Psi(z)\ \xi\mu J_\nu\ (\xi)$$

$$y_2(z) \equiv \Psi(z)\ \xi\mu Y_\nu(\xi)$$

where the order of the Bessel function ν is determined by 1/2 A. For complex number, it is advantageous to express the solution of Equation (64) in terms of Hankel functions as follows:

$$\underline{Z}_1 = \frac{\pi^{1/2}}{2} \left\{ \frac{\chi}{(1 - t^2)^{1/2}} \right\}^{-1/2} \Phi^{1/4}\ \rho^{1/4}\ e^{i\ 3/4\pi}\ H_1^{(i)}\ (\rho) + c \quad (66)$$

$$\underline{Z}_2 = \frac{\pi^{1/2}}{2} \left\{ \frac{\chi}{(1 - t^2)^{1/2}} \right\}^{1/2} \Phi^{1/4}\ \rho^{1/4}\ e^{-i\ 3/4\pi}\ H_1^{(2)}\ (\rho) + c \quad (67)$$

where

$$H_1^{(1)}\ (\rho) = \frac{J_{-1}\ (\rho) - e^{-\pi i}\ J_1\ (\rho)}{i \sin \pi}$$

$$= \frac{Y_{-1}\ (\rho) - e^{-\pi i}\ Y_1\ (\rho)}{\sin \pi}$$

$$H_1^{(2)}\ (\rho) = \frac{e^{\pi i}\ J_1\ (\rho) - J_{-1}\ (\rho)}{i \sin \pi}$$

$$= \frac{Y_{-1}\ (\rho) - e^{\pi i}\ Y_1\ (\rho)}{\sin \pi}$$

Both $H_1^{(1)}(\rho)$ and $H_1^{(2)}(\rho)$ are the third kind, first order Bessel function (see reference [9]),

$$\rho = (8i^3 \mu^2)^{1/2} \Phi \qquad (68)$$

$$\Phi = \int_0^t \frac{\chi}{(1 - t^2)^{1/2}} dt \qquad (69)$$

In Equations (66) and (67)

$$c = \left[\frac{\chi}{(1 - t^2)^{1/2}} \right]^{-1/2} \Phi^{1/4} \rho^{-3/4} \frac{\log(2i^3 \mu^2)^{1/2}}{(2i^3 \mu^2)^{1/2}} \mathcal{O}(\epsilon) \qquad (70)$$

and $\mathcal{O}(\epsilon)$ denotes a bounded function.

In view of Equations (66), (67) and (70), it follows that the homogeneous solution of Equation (64) may be represented asymptotically by

$$\underline{Z}_H = \left[\frac{\chi}{(1 - t^2)^{1/2}} \right]^{-1/2} \Phi^{1/4} \rho^{1/4} \left\{ \underline{A}\, H_1^{(1)}(\rho) + \underline{B}\, H_1^{(2)}(\rho) \right\} \qquad (71)$$

which is valid in $|t| < 1$ and where the constant $\underline{A}$ and $\underline{B}$ are complex.

It is shown in [10] that the first and third kinds of Bessel functions whose arguments have their phases equal to $(1/4)\pi$ or $(3/4)\pi$, are equivalent to

$$\mathrm{ber}_n(q) \pm i\,\mathrm{bei}_n(q) = J_n(q e^{\pm 3/4 \pi i})$$

$$\mathrm{her}_n(q) \pm i\,\mathrm{hei}_n(q) = H_n^{(1)}(q e^{\pm 3/4 \pi i})$$

and

$$\text{ker}\ (q) = (-1/2)\,\pi\ \text{hei}\ (q)\ ,$$

$$\text{kei}\ (q) = (1/2)\ \pi\ \text{her}\ (q)\ .$$

where $\text{ber}_n\ (q)$ and $\text{bei}_n(q)$ are real and imaginary parts of $J_n\ (qe^{\pm\ 3/4\ \pi\, i})$ respectively etc.

Using the above expressions, Equation (71) may be expressed as follows:

$$\underline{Z}_H = \left[\frac{\chi}{(1 - t^2)^{1/2}}\right]^{-1/2} \Phi^{1/2} \left\{ \underline{A}_1[\text{ber}_1(q) + i\ \text{bei}_1\ (q)] + \underline{B}_1\ [\text{ker}_1\ (q) + i\ \text{kei}_1\ (q)] \right\} \qquad (72)$$

where

$$q = (8\mu^2)^{1/2}\ \Phi \qquad (73)$$

and

$$\underline{A}_1 = A_o + iA_1\ , \qquad (74)$$

$$\underline{B}_1 = B_o + iB_1$$

Transforming Equation (52) by means of

$$t = \sin\ (\varphi/2)$$

it becomes

$$Z = [1 - 4t^2(1 - t^2)]^{1/2}\ (2t)^{1/2}\ (1 - t^2)^{1/4}\ (\beta + i\ ) \qquad (75)$$

By means of Equations (75) and (57) and solution Equation (72), the expressions of β, ψ, β', and ψ' may be written as

$$\left\{2t\left[1 - 4t^2 (1 - t^2)\right]\right\}^{1/2} (1 - t^2)^{1/4} (\beta + i\chi)$$

$$= \frac{1}{(1 - t^2)^{1/4}} \left[\frac{\chi}{(1 - t^2)^{1/2}} \right]^{-1/2} \Phi^{1/2} \cdot$$

$$\left\{ \begin{bmatrix} A_o \,\mathrm{ber}_1(q) + B_o \,\mathrm{ker}_1(q) \\ -A_1 \,\mathrm{bei}_1(q) - B_1 \mathrm{kei}_1(q) \end{bmatrix} + i \begin{bmatrix} A_o \,\mathrm{bei}_1(q) + A_1 \mathrm{ber}_1(q) \\ B_o \,\mathrm{kei}_1(q) + B_1 \mathrm{kei}_1(q) \end{bmatrix} \right\}$$

$$\beta = \eta \left\{A_o \mathrm{ber}\ (q) - A_1 \mathrm{bei}\ (q) + B_o \mathrm{ker}\ (q) - B_1 \mathrm{kei}\ (q)\right\} \tag{76}$$

$$\psi = \eta \left\{A_o \mathrm{bei}\ (q) + A_1 \mathrm{ber}\ (q) + B_o \mathrm{kei}\ (q) + B_1 \mathrm{ker}\ (q)\right\} \tag{77}$$

$$\beta' = \mathcal{L} \beta + \frac{\chi}{(1 - t^2)^{1/2}} (8\mu^2)^{1/2} \eta \cdot$$

$$\begin{Bmatrix} A_o \mathrm{ber}'\ (q) - A_1 \mathrm{bei}'\ (q) \\ B_o \mathrm{ker}'\ (q) - B_1 \mathrm{kei}'\ (q) \end{Bmatrix} \tag{78}$$

$$\psi' = \mathcal{L} \psi + \frac{\chi}{(1 - t^2)^{1/2}} (8\mu^2)^{1/2} \eta \cdot$$

$$\begin{Bmatrix} A_o \mathrm{bei}'\ (q) + A_1 \mathrm{ber}'\ (q) \\ A_o \mathrm{kei}'\ (q) + B_1 \mathrm{ker}'\ (q) \end{Bmatrix} \tag{79}$$

where

$$\mathcal{L} = \frac{1}{2(1 - t^2)^{1/2}} \frac{\chi}{\Phi} - \frac{1}{2t} - \frac{t\ (1 - 2t)}{1 - 4t^2(1 - t^2)}$$

$$- \frac{t}{2\ (1 - t^2)} \tag{80}$$

$$\eta = \left(\frac{\Phi}{2t}\right)^{1/2} \left[1 - 4t^2\,(1 - t^2)\right]^{-1/2} \left[\frac{\chi}{(1 - t^2)^{1/2}}\right]^{-1/2} \qquad (81)$$

CHAPTER IV

NUMERICAL RESULTS

Introduction

In this chapter, some applications to specific problems of the general results obtained in Chapter III and the comparison of the solution with shallow spherical shells are considered.

Based upon the solution given in Chapter III the explicit results for the following two problems are obtained:

1. Uniform stress couple M_α applied around the edge $\varphi = \alpha$; as shown in Figure 4;
2. Uniform radial stress resultant H_α applied around the edge $\varphi = \alpha$; as shown in Figure 5.

Also, the problem of a shell with no edge restraint carrying a load uniformly distributed over the whole area is solved by using membrane theory as shown in Figure 6.

Paraboloidal Shells under Edge Loadings

In this section are considered paraboloidal shells (closed at the apex $\varphi = 0$) under only edge loadings as follows: (1) uniform stress couple M_α applied around the edge $\varphi = \alpha$, and (2) uniform radial stress resultant H_α applied around the edge $\varphi = \alpha$.

In both examples under consideration, the transition conditions at $\varphi = 0$ require that

$$\varphi = 0;\ \beta, \psi, \beta', \psi', \text{ remain finite.}$$

The constant coefficients of the functions ker (q) and kei (q) are set equal to zero at the outset.

Example (1): The boundary conditions in this case are

when $\varphi = \alpha;$ $M_\varphi = M_\alpha, \quad Q = 0$ (82)

or, equivalently,

$$\varphi = \alpha; \qquad \beta = \frac{r_1}{D} M_\alpha - \nu \frac{r'}{r} \beta, \quad \psi = 0 \tag{83}$$

where α is the value of φ at the edge of the shell. Since no particular solution is needed for the unloaded region, $\beta = \beta_H$ (β_H is the homogeneous solution of β).

Applying the above conditions to solutions Equations (76), (77) and (78), the constants A_o and A_1 are determined as follows:

$$A_o = \frac{A}{D} M_\alpha \frac{\text{ber } (\bar{q})}{\cos \alpha} \cdot \eta \tag{84}$$

$$A_1 = - \frac{A}{D} M_\alpha \frac{\text{bei } (\bar{q})}{\cos \alpha} \cdot \eta \tag{85}$$

where

$$\eta = \mathcal{m}^{-1} \quad (\mathcal{L} + \nu \frac{\sec \alpha}{\sin \alpha}) \left\{ \text{ber}^2 (\bar{q}) + \text{bei}^2 (\bar{q}) \right\}$$

$$+ \frac{\gamma}{(1 - t^2)^{1/2}} (8\mu^2)^{1/2}$$

$$[\text{ber}' (\bar{q}) \text{ber } (\bar{q}) + \text{bei}' (\bar{q}) \text{ bei } (\bar{q})]^{-1} \tag{86}$$

and

$$\bar{q} = q \ (\alpha)$$

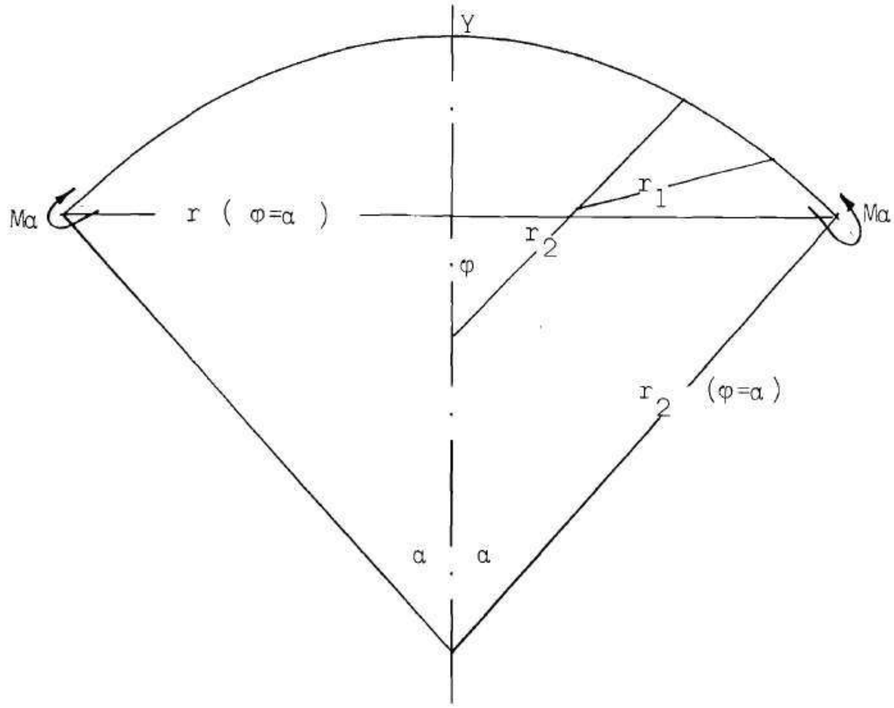


Figure 4. Paraboloidal Shell under Uniform Stress Couples M_{α} Applied Around the Edge.

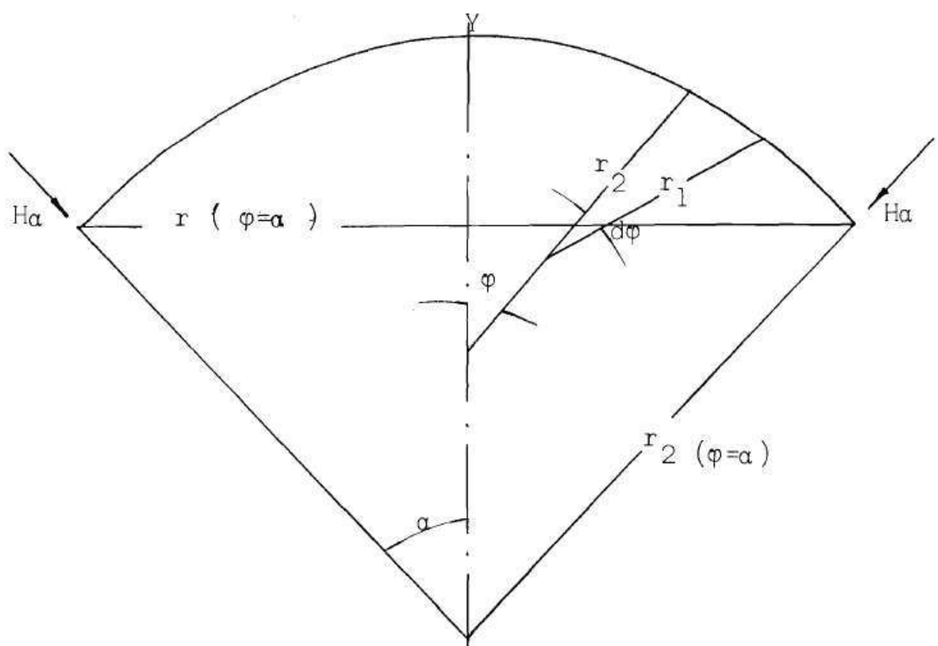


Figure 5. Paraboloidal Shell under Uniform Radial Stress Resultant H_{α} Applied Around the Edge.

The primes in these equations denote differentiation with respect to q.

Example (2): With boundary conditions

$$\varphi = \alpha, \qquad Q = -H_\alpha, \qquad M_\varphi = 0 \tag{87}$$

$$\varphi = \alpha, \qquad \psi = \frac{mr}{Eh^2} H_\alpha, \qquad \beta' = \nu \frac{r'}{r} \beta \tag{88}$$

and proceeding as in the previous example, the constants A_o and A_1 are

$$A_o = \eta \frac{mr}{Eh^2} H_\alpha \left\{ \left(\mathcal{L} + \nu \frac{r'}{r}\right) \text{bei}\ (\bar{q}) + \frac{\chi}{(1 - t^2)^{1/2}} (8\mu^2)^{1/2}\ \text{bei}'\ (\bar{q}) \right\} \tag{89}$$

$$A_1 = \eta \frac{mr}{Eh^2} H_\alpha \left\{ \left(\mathcal{L} + \nu \frac{r'}{r}\right) \text{ber}\ (\bar{q}) + \frac{\chi}{(1 - t^2)^{1/2}} (8\mu^2)^{1/2}\ \text{ber}'\ (\bar{q}) \right\} \tag{90}$$

Basic Equations of Paraboloidal Shells under Uniform Pressure with No Edge Restraint

According to Timoshenko [6], in the membrane theory of shells, the bending stresses of shells is neglected and only the stresses due to strain in the middle surface of the shell need be considered. From the assumed asymmetry of loading and deformation it can be concluded that there will be no shearing forces acting on the sides of the element. The magnitudes of the normal forces per unit length are denoted by N_φ and N_ξ as shown in Figure 7.

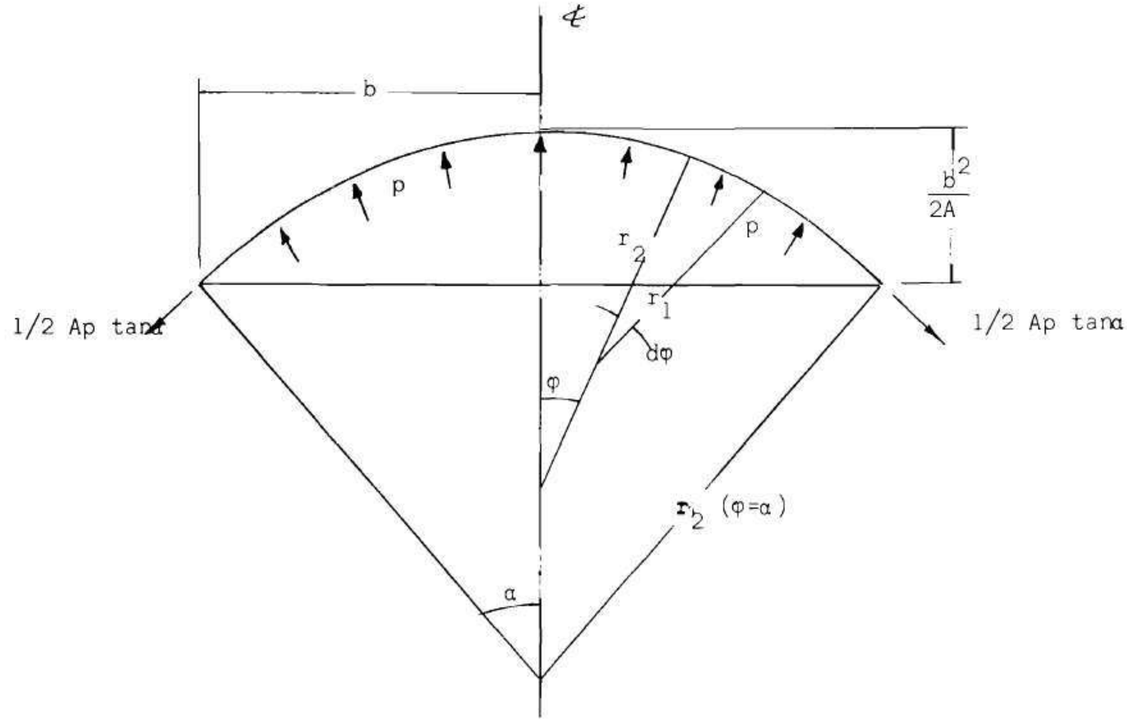


Figure 6. Paraboloidal Shell under Uniform Pressure p with Simple Support at Edge.

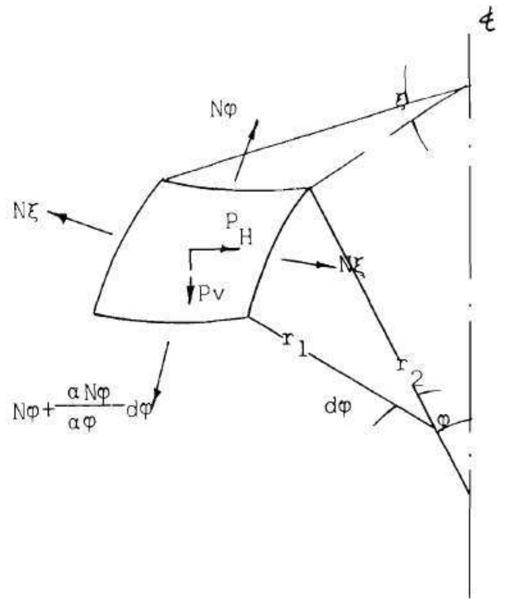


Figure 7. An Element of Shell Loaded Symmetrically with Respect to the Geometric Axis. Also Shown are Membrane Stresses and Load Intensity Components.

Assume that a paraboloidal shell is submitted to the action of a uniform pressure, the magnitude of which per unit area is constant and equal to p as shown in Figure 6. Denote the equation of the middle surface in rectangular Cartesian coordinates by

$$X^2 + Z^2 - b^2 = a\ Y$$

In view of the presence of a distributed load, as in [6], a suitable particular solution may be obtained approximately by the membrane theory of shells.

The basic equations of paraboloidal shells in this case may be written as:

$$R = -A^2 \pi p \tan^2 \varphi \tag{91}$$

$$N_\varphi = 1/2\ PA \cos \varphi \tag{92}$$

$$N_\xi = Ap\ (\sec \varphi - 1/2 \cos \varphi) \tag{93}$$

$$v = \sin \varphi \left\{ (pA^2/Eh)\ [1/3\ (1/2 - \nu)\ \sec^3 \varphi - \frac{1}{2} \sec \varphi] + c \right\} \tag{94}$$

$$u = \cos \varphi \left\{ (pA^2/Eh)\ \Phi + c - (pA^2/Eh)(1 - \frac{1}{2}\nu)\sec^2\varphi - \frac{1}{2} \right\} \tag{95}$$

where

$$\Phi = (1/3)\ (1/2 - \nu)\ \sec^3 \varphi - \frac{1}{2} \sec \varphi \tag{96}$$

R is the resultant of the total load on that portion of the shell; the other notations used here are the same as mentioned before.

Example (3): The boundary conditions in this case are

$$\text{when} \quad \varphi = \alpha\ ; \qquad v = 0 \tag{97}$$

Applying the above conditions to the solution of Equation (94), the constant c is determined as follows:

$$c = -(pA^2/Eh)\Phi \ . \tag{98}$$

Comparison of the Solution with a Shallow Spherical Shell

According to E. Reissner [11], a segment will be called shallow if its height to base diameter is less than, say, 1/8. The results obtained on the basis of this assumption will often also be applicable to shells which are not shallow, namely when the loads are such that the stresses are effectively restricted to shallow zones. Here, in a paraboloidal shell, whenever φ is very small, say $\pi/8$, the meridian curve in this small region is equivalent to a shallow spherical shell. This can be shown as follows:

$$\frac{dy}{dr} << 1 \tag{99}$$

and the equation of paraboloidal shell is $r^2 - b^2 = 2Ay$

$$1/r_2 = \frac{d^2y/dx^2}{[1 + (dy/dr)^2]^{3/2}} \approx \frac{d^2y}{dr^2} = \frac{1}{A} = \text{const.} \tag{100}$$

Equation (99) also represents radius of the principal curvature of the shallow spherical shell may be written as

$$r = \frac{A^3 \operatorname{Tan}^2 \varphi'}{(A^2 \tan^2 \varphi') - b^2} \tag{101}$$

where φ' is the value of φ smaller than $\pi/8$.

For the purpose of comparing the solution of a paraboloidal shell to the equivalent shallow spherical shells, it seems necessary first to reduce the solution to a shallow shell. When φ is very small, say $\pi/8$, the solutions of Equations (76), (77), (78), (79), (84), (85), (89) and (90) may be written as

$$\beta = \mathcal{M}' \begin{Bmatrix} A_o \text{ ber } (q) - A_1 \text{ bei } (q) \\ B_o \text{ ker } (q) - B_1 \text{ kei } (q) \end{Bmatrix} \tag{102}$$

$$\psi = \mathcal{M}' \begin{Bmatrix} A_o \text{ bei } (q) + A_1 \text{ ber } (q) \\ B_o \text{ kei } (q) + B_1 \text{ ker } (q) \end{Bmatrix} \tag{103}$$

$$\beta' = \mathcal{L}' \beta + \chi (8\mu^2)^{1/2} \mathcal{M}' \begin{Bmatrix} A_o \text{ ber}' (q) - A_1 \text{ bei}' (q) \\ B_o \text{ ker}' (q) - B_1 \text{ kei}' (q) \end{Bmatrix} \tag{104}$$

$$\psi' = \mathcal{L}' \psi + \chi (8\mu^2)^{1/2} \mathcal{M}' \begin{Bmatrix} A_o \text{ bei}' (q) + A_1 \text{ ber}' (q) \\ B_o \text{ kei}' (q) + B_1 \text{ ker}' (q) \end{Bmatrix} \tag{105}$$

$$A_o = (A/D)\ M_\alpha\ (\text{ber } (\bar{q})/\cos \varphi') \cdot \eta' \tag{106}$$

$$A_1 = -(A/D)\ M_\alpha\ (\text{bei}(\bar{q})/\cos \varphi') \cdot \eta' \tag{107}$$

$$A_o = \eta' (mr/Eh^2)\ H_\alpha \left\{ (\mathcal{L}' + \nu \frac{r'}{r}) \text{ bei } (\bar{q}) + \chi (8\mu^2)^{1/2} \text{bei}'(\bar{q}) \right\} \tag{108}$$

$$A_1 = \eta' (mr/Eh^2)\ H_\alpha \left\{ (\mathcal{L}' + \nu \frac{r'}{r}) \text{ ber } (\bar{q}) + \chi (8\mu^2)^{1/2} \text{ ber}'(\bar{q}) \right\} \tag{109}$$

respectively, where

$$\mathcal{L}' = (1/2)\ (\chi/\Phi) - (1/2t) - (3t/2) \tag{110}$$

$$\mathcal{M}' = (\Phi/2t)^{1/2}\, \chi^{-1/2} \tag{111}$$

$$\eta' = \mathcal{M}^{-1} \Big\{ (\alpha' + \nu \frac{r'}{r}) \left[\mathrm{ber}^2 (\bar{q}) + \mathrm{bei}^2 (\bar{q}) \right] + \chi\, (8\mu^2)^{1/2} \left[\mathrm{ber}'(\bar{q})\, \mathrm{ber}\, (\bar{q}) + \mathrm{bei}'\, (\bar{q})\, \mathrm{bei}\, (\bar{q}) \right] \Big\} \tag{112}$$

Then, as mentioned in Timoshenko [6], the approximate method for the spherical shells is the method of asymptotic integration, due to O. Blumenthal [7], the solutions may be written as follows:

$$N_\varphi = - \cot\, (\alpha - \chi) C e^{-\lambda\chi} \sin\, (\lambda\chi + \gamma) \tag{113}$$

$$N_\xi = - \lambda\, \sqrt{2} \cdot C e^{-\lambda\chi} \sin\, (\lambda\chi + \gamma - \tfrac{\pi}{4}) \tag{114}$$

$$V = -(1/Eh)\; 2\lambda^2\, C e^{-\lambda\chi} \cos\, (\lambda\chi + \gamma) \tag{114}$$

$$M_\varphi = (r/\lambda\, \sqrt{2})\; C e^{-\lambda\chi} \sin\, (\lambda\chi + \gamma + \tfrac{\pi}{4}) \tag{115}$$

$$M_\xi = (r\nu/\lambda\, \sqrt{2}\,)\; C e^{-\lambda\chi} \sin\, (\lambda\chi + \gamma + \tfrac{\pi}{4}) \tag{116}$$

$$\delta = - (r/Eh) \sin\, (\alpha - \chi)\; \lambda\, \sqrt{2}\; C e^{-\lambda\chi} \sin(\lambda\chi + \gamma - \tfrac{\pi}{4}) \tag{117}$$

where C and γ are constants; and $\chi = \alpha - \varphi$ as shown in Figure 8. If the spherical shell is under uniform stress couple along the edge $\varphi = \alpha$ only as shown in Figure 9, according to the boundary conditions. When $\varphi = \alpha$;

$$M_\varphi = M_\alpha, \qquad N_\varphi = 0 \tag{118}$$

the constants C and γ become

$$C = M_\alpha 2\lambda/r \tag{119}$$

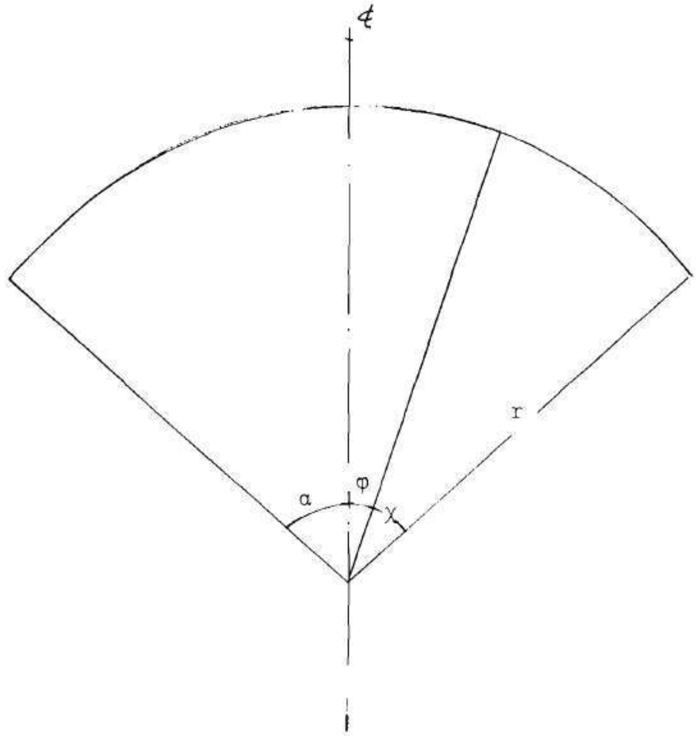


Figure 8. Relationship Between α, φ and for Shallow Spherical Shell.

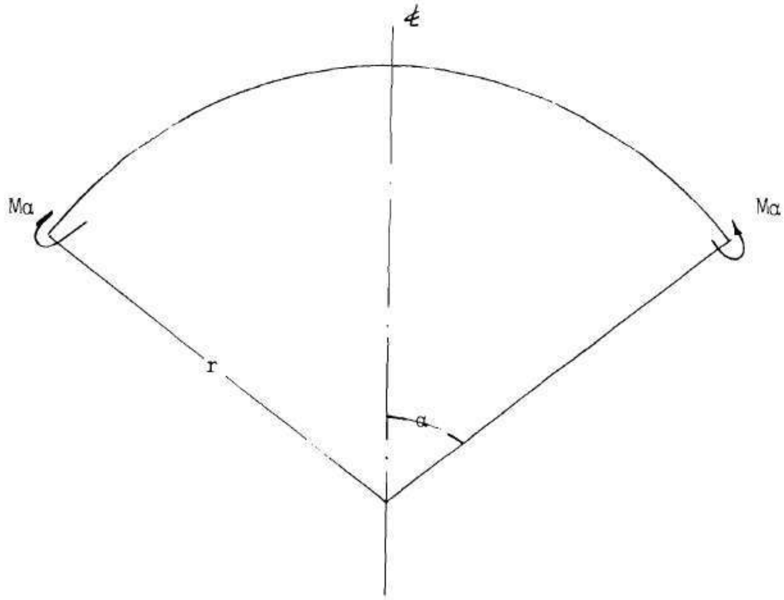


Figure 9. Shallow Spherical Shell under Uniform Stress Couple M_α along the Edge $\varphi = \alpha$.

$$r = 0 \tag{120}$$

where

$$\lambda^4 = 3\ (1 - \nu^2)(r/h)^2 \tag{121}$$

By using the solutions in Example (2) and the solutions of Equations (113) to (121), the comparison of the solution with spherical shells may be shown in Figure 10.

Some Properties of the Functions ber, bei, ker, kei

Tables of these functions and their derivatives are in existence [12] and have been used in what follows.

For small argument values

$$\text{ber}\ (q) = 1 - q^4/2^2 \cdot 4^2 + \ldots\ldots\ldots\ldots \tag{122}$$

$$\text{bei}\ (q) = q^2/2 - q^6/q^2{\cdot}4^2{\cdot}6^2 + \ldots\ldots\ldots \tag{123}$$

$$\text{ker}\ (q) = -\ln q + .1159 + \pi q^2/16 + \ldots\ldots\ldots \tag{124}$$

$$\text{kei}\ (q) = -(q^2/4)\ln x - \pi/4 + 1.1159\ q^2/4 + \ldots\ldots\ldots \tag{125}$$

For large argument values

$$\text{ber}\ (q) = \frac{\exp\ (q/\sqrt{2})}{(2\pi q)^{1/2}}\ \cos\ (q/\sqrt{2} - \pi/8) \tag{126}$$

$$\text{bei}\ (q) = \frac{\exp\ (q/\sqrt{2})}{(2\pi q)^{1/2}}\ \sin\ (q/\sqrt{2} - \pi/8) \tag{127}$$

$$\text{ker}\ (q) = \frac{\exp\ (-q/\sqrt{2})}{(2q/\pi)^{1/2}}\ \cos\ (q/\sqrt{2}) + \pi/8) \tag{128}$$

$$\text{kei}\ (q) \cong \frac{\exp\ (-q/\sqrt{2})}{(2q/\pi)^{1/2}}\ \sin\ (q/\sqrt{2} - \pi/8) \tag{129}$$

For recurrence formulas

$$\text{ber}_1\ (q) = (1/\sqrt{2})\ (\text{ber}'\ (q) - \text{bei}'\ (q)) \tag{130}$$

$$\text{bei}_1\ (q) = (1/\sqrt{2})\ (\text{ber}'\ (q) + \text{bei}'\ (q)) \tag{131}$$

$$\begin{aligned} \text{ber}_n'\ (q) &= -\ (1/\sqrt{2})\ (\text{ber}_{n-1}(q) + \text{bei}_{n-1}(q)) \\ &\quad -\ (n\ \text{ber}_n\ (q)/q) \end{aligned} \tag{132}$$

$$\begin{aligned} \text{bei}_n'\ (q) &= (1/\sqrt{2})\ (\text{ber}_{n-1}\ (q) - \text{bei}_{n-1}\ (q)) \\ &\quad -\ (n\text{bei}_n(q)/q) \end{aligned} \tag{133}$$

Numerical Solutions

By using [13] and the properties of the functions ber, bei, ker, kei, the components of stress were obtained for the case of

$$a/b = 2, \quad a/h = 20, \quad \text{and}\ \nu = 0.3$$

of Example (1) and (2) as shown in Figure 11 and Figure 12 respectively.

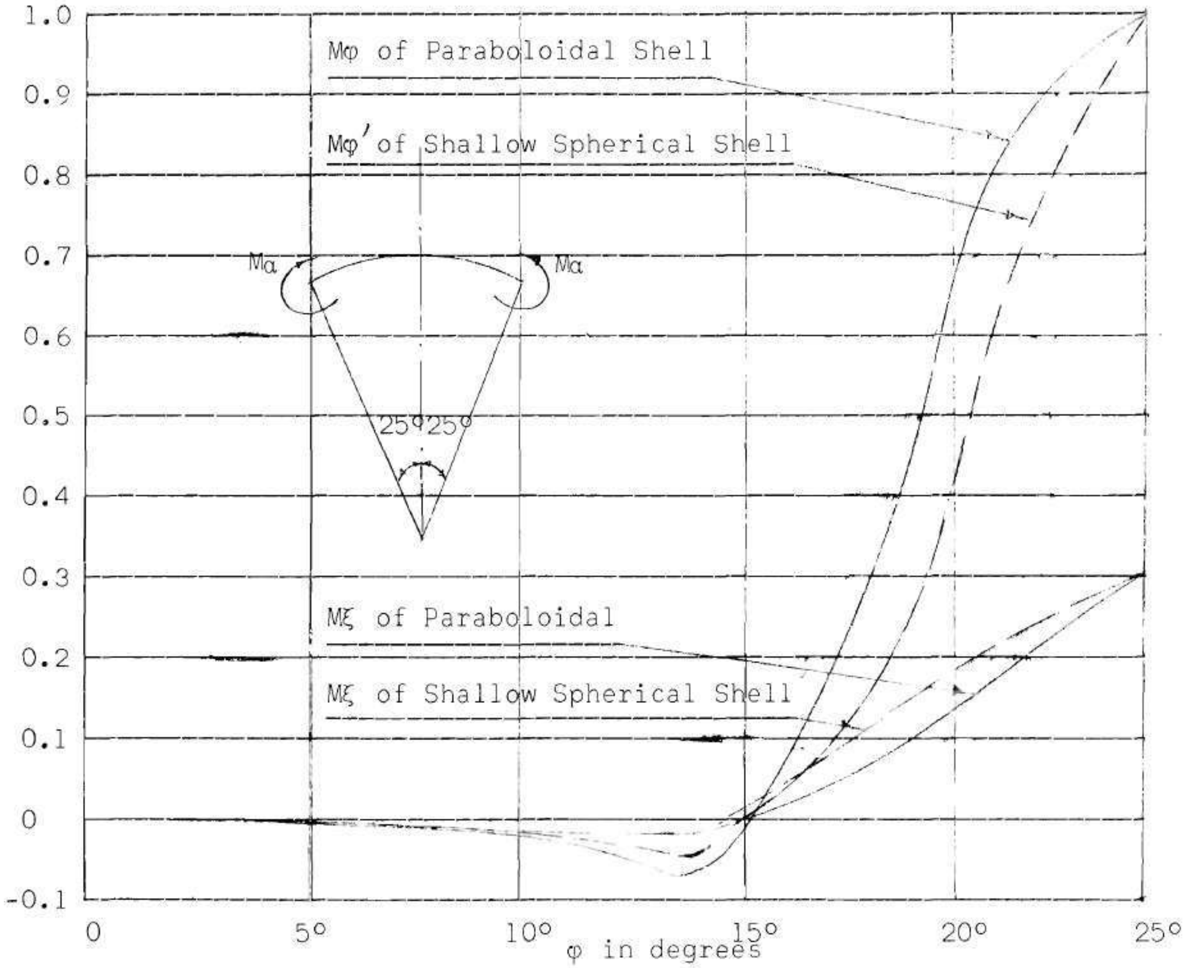
Conclusion

The membrane solution can be obtained without much difficulty, however, this solution in general can not satisfy the kinematic boundary conditions along the edge of the shell. Moment and transverse shear must be considered here in order to enforce compatible boundary conditions.

By superposing the solutions of Example (1) or Example (2) or both on the membrane solution, the stress distribution and deformation for paraboloidal shells of revolution under axisymmetric loads with any type of edge restraints can readily be obtained.

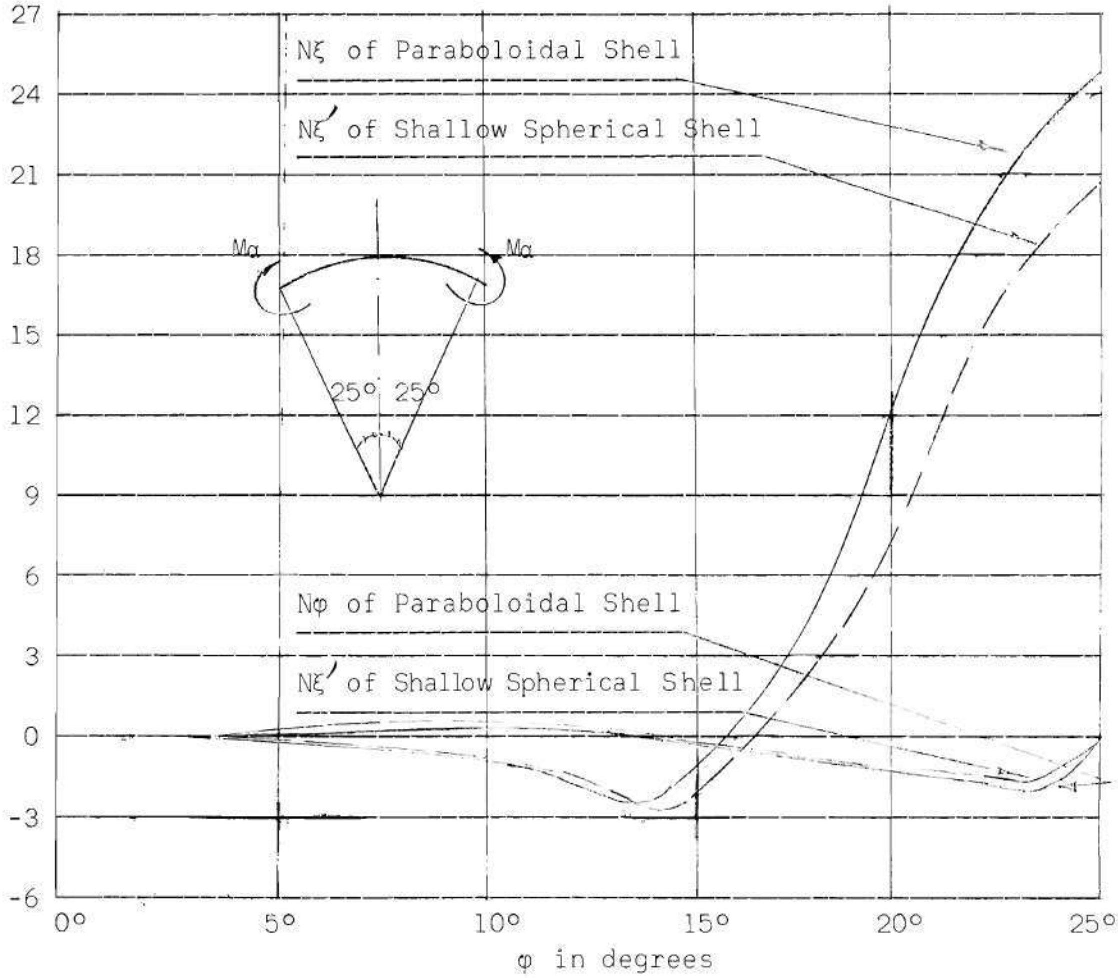
Discussion of Numerical Results

It has been mentioned in Timoshenko [6], that the case in which the angle φ is small the solution of Equations (113) to (116) are not sufficiently accurate. When a comparison is made between the solution in Figure 10(a) and (b) with the solution in Figure 186 and Figure 187 in [6], it is obvious that the solution in Figure 10(a) and (b) is closer to the exact solution than the solution of Equations (113) to (116) when φ is small.



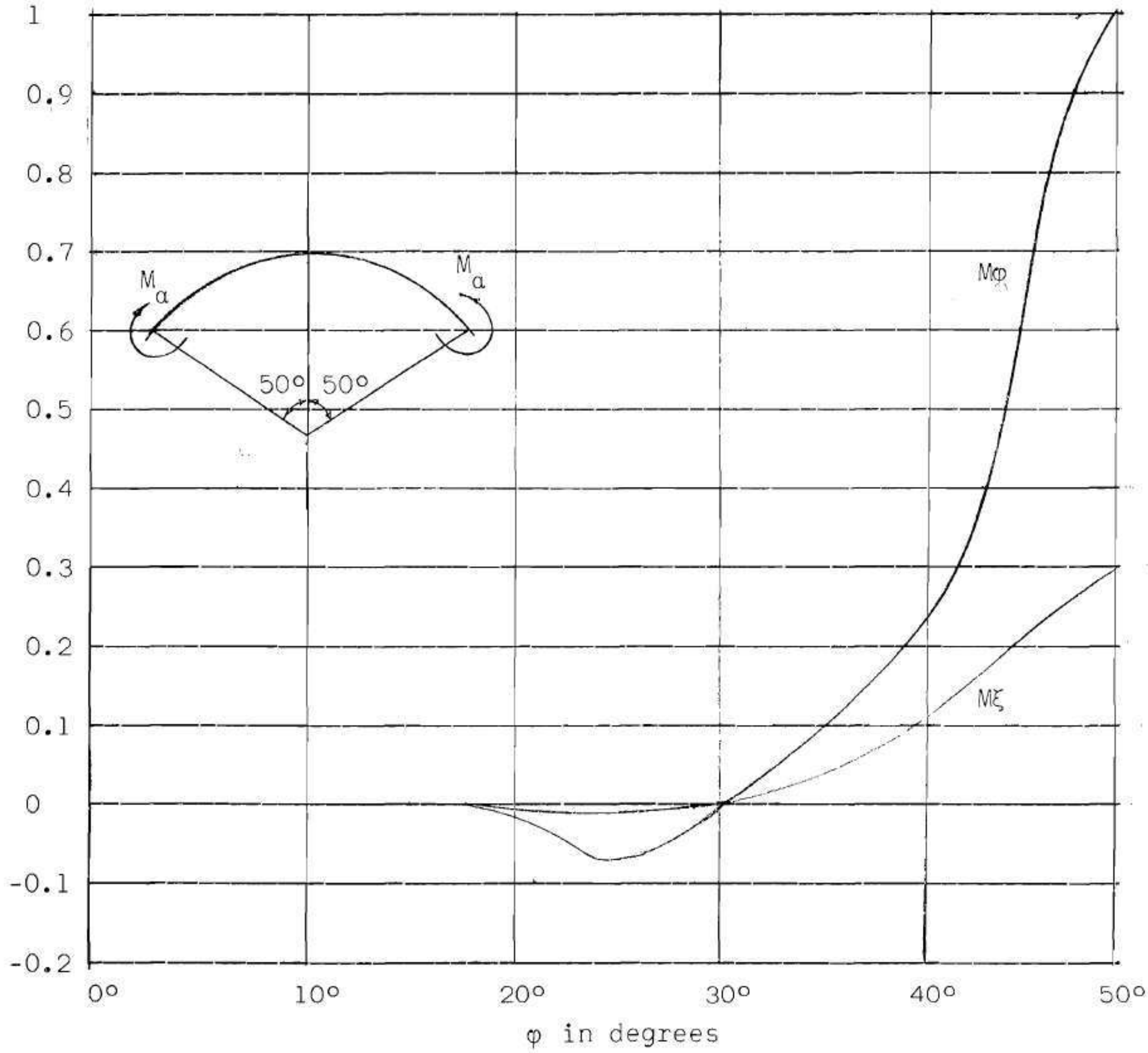
	25°	20°	15°	10°	5°
$M\varphi$	M_α	$.632\ M_\alpha$	$-.117\ M_\alpha$	$-.088\ M_\alpha$	$-.004\ M_\alpha$
$M\varphi$	M_α	$.375\ M_\alpha$	$.008\ M_\alpha$	$-.084\ M_\alpha$	0
$M\xi$	$.3\ M_\alpha$	$.129\ M_\alpha$	$-.005\ M_\alpha$	$-.026\ M_\alpha$	0
$M\xi'$	$.306\ M_\alpha$	$.170\ M_\alpha$	$+.0026\ M_\alpha$	$-.025\ M_\alpha$	0

Figure 10(a). Moment Diagram in a Paraboloidal Shell and the Equivalent Shallow Spherical Shell ($r = .276A$) due to an Uniform Stress Moment M_α along the Edge $\varphi = 25°$, ($\nu = 0.3$, $A = 1/2\ a$, $a/b = 2$, $a/h = 20$).



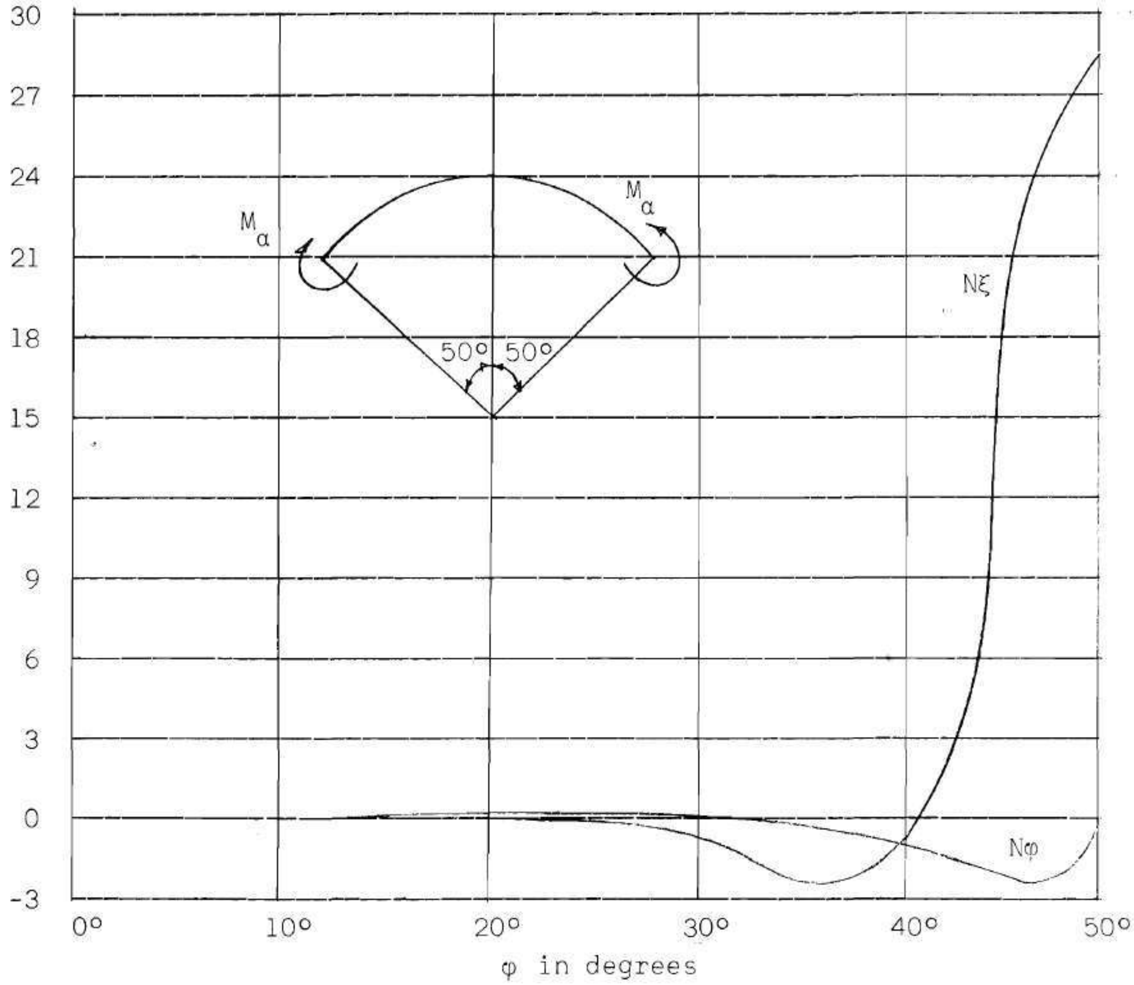
	25°	20°	15°	10°	5°
Nξ	$24.46M_\alpha/A$	$10.90M_\alpha/A$	$-1.8M_\alpha/A$	$-.65M_\alpha/A$	$.071M_\alpha/A$
Nξ′	$20.00M_\alpha/A$	$6.71M_\alpha/A$	$-2.6M_\alpha/A$	$-.54M_\alpha/A$	$.049M_\alpha/A$
Nφ	0	$-.64M_\alpha/A$	$-.512M_\alpha/A$	$.007M_\alpha/A$	0
Nφ′	0	$-.45M_\alpha/A$	$-.11\ M_\alpha/A$	$.014M_\alpha/A$	0

Figure 10(b). Normal Stresses Distribution in a Paraboloidal Shell ($a/b = 2$, $a/h = 20$) and the Equivalent Shallow Spherical Shell ($r = .276A$, $A = 1/2\ a$) due to an Uniform Stress Moment M_α Along the Edge $\varphi = 25°$, ($\nu = 0.3$).



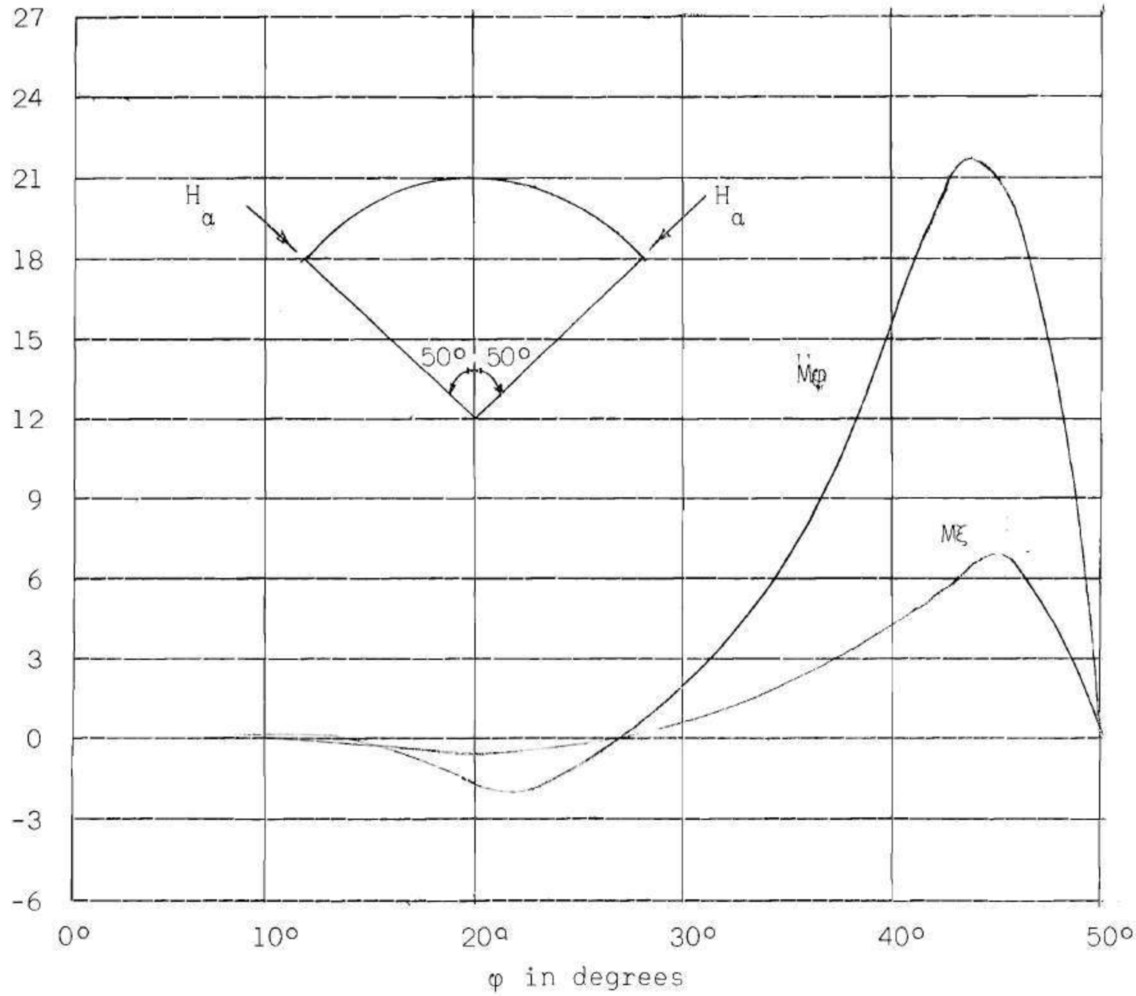
	50°	45°	40°	30°	25°	20°
M_φ	M_α	.544 M_α	.261 M_α	-.0087 M_α	-.0754 M_α	-.0115 M_α
M_ξ	.303 M_α	.183 M_α	.114 M_α	-.0022 M_α	-.0137 M_α	-.0036 M_α

Figure 11(a). Moment Diagram in a Paraboloidal Shell ($a/b = 2$, $a/h = 20$) due to an Uniform Stress Moment M_α Along the edge $\varphi = 50°$ ($\nu = 0.3$).



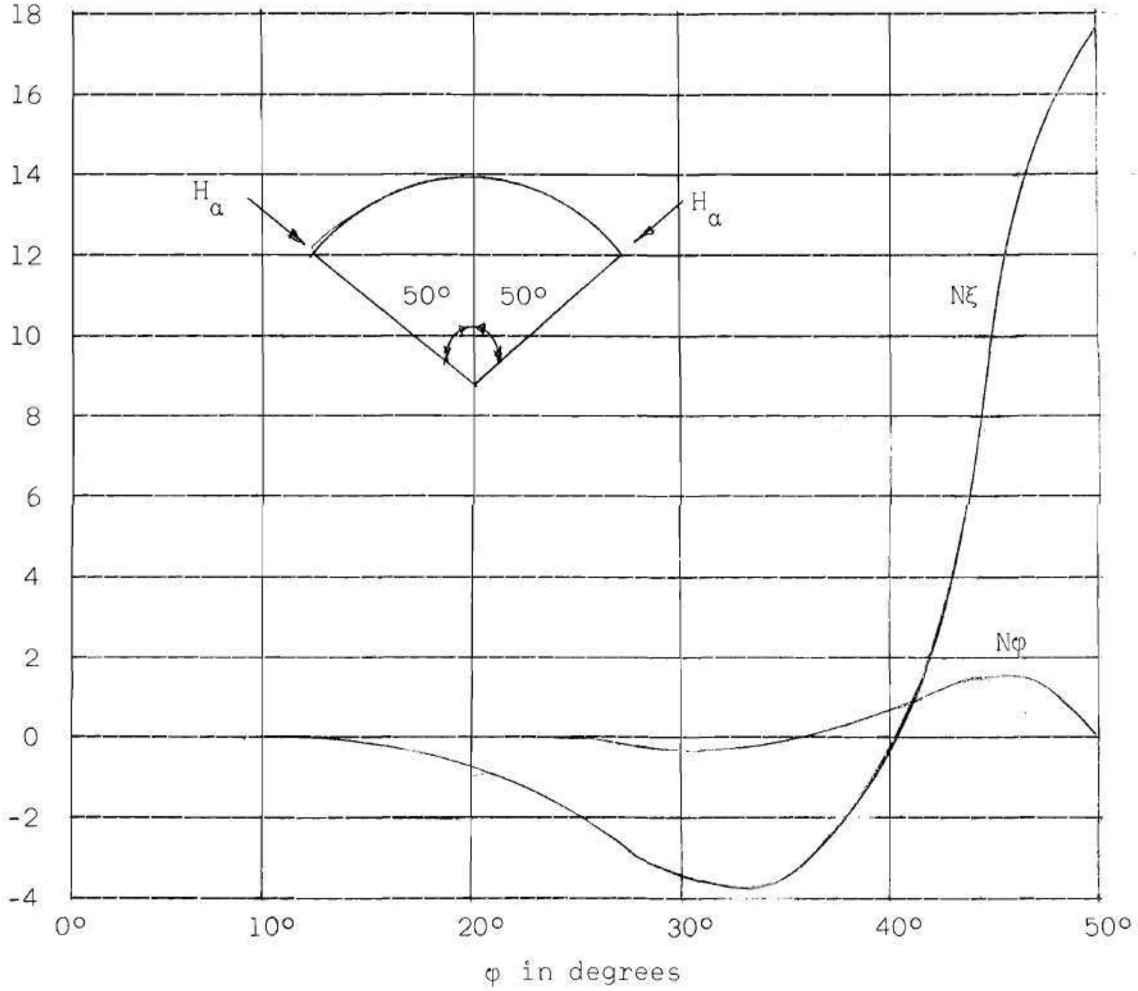
	50°	45°	40°	30°	25°	20°
N_φ	0	$-1.87M_\alpha/A$	$-.557M_\alpha/A$	$-.099M_\alpha/A$	$-.0825M_\alpha/A$	$.00114M_\alpha/A$
N_ξ	$28.4M_\alpha/A$	$6.22M_\alpha/A$	$-1.69M/A$	$-0.981M_\alpha/A$	$-0.66M_\alpha/A$	$.256\ M_\alpha/A$

Figure 11(b). Normal Stresses Distribution in a Paraboloidal Shell ($a/b = 2$, $a/h = 20$) due to an Uniform Stress Moment M_α Along the Edge $\varphi = 50°$ ($\nu = 0.3$).



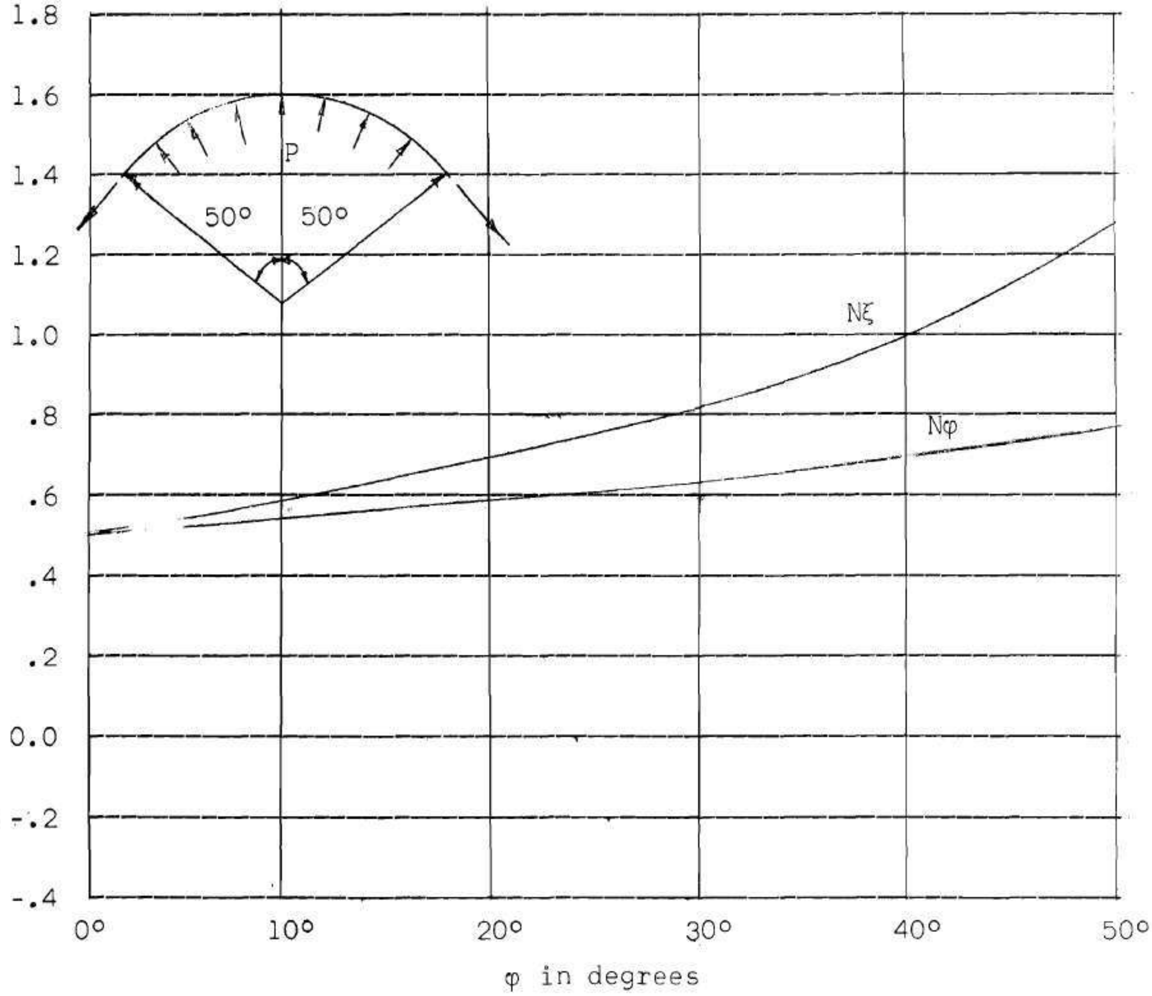
	50°	45°	40°	30°	25°	20°
M_{φ}	0	$22.1AH_{\alpha}$	$13.11AH_{\alpha}$	$1.76AH_{\alpha}$	$-0.143AH_{\alpha}$	$-0.877AH_{\alpha}$
M_{ξ}	$0.0014AH_{\alpha}$	$6.62AH_{\alpha}$	$3.42AH_{\alpha}$	$0.55AH_{\alpha}$	$-0.081AH_{\alpha}$	$-0.0057AH_{\alpha}$

Figure 12(a). Moment Diagram in a Paraboloidal Shell ($a/b = 2$, $a/h = 20$) due to an Uniform Radial Stress Resultant H_{α} Along the Edge $\varphi = 50°$ ($\nu = 0.3$).



	50°	45°	40°	30°	25°	20°
N_φ	$.036H_\alpha$	$.643H_\alpha$	$.069H_\alpha$	$-.065H_\alpha$	$-.127H_\alpha$	$-.053H_\alpha$
N_ξ	$17.6H_\alpha$	$11.0\ H_\alpha$	$-.25\ H_\alpha$	$-3.03H_\alpha$	$-.96\ H_\alpha$	$-.325H_\alpha$

Figure 12(b). Normal Stresses Distribution in a Paraboloidal Shell ($a/b = 2$, $a/h = 20$) due to an Uniform Radial Stress Resultant H_α Along the Edge $\varphi = 50$ ($\nu = 0.3$).



	50°	45°	35°	25°	15°	0°
N_φ	.778 pA	.707 pA	.61 pA	.552 pA	.518 pA	.5 pA
N_ξ	1.234 pA	1.060 pA	.81 pA	.65 pA	.545 pA	.5 pA

Figure 13. Normal Stresses Distribution in a Paraboloidal Shell ($a/b = 2$, $a/h = 20$) due to an Uniform Pressure p with No Edge Restraint ($\nu = 0.3$).

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